

Back Reaction Of Perturbations In Two Scalar Field Inflationary Models

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We calculate the back-reaction of long wavelength cosmological perturbations on a general relativistic measure of the local expansion rate of the Universe. Specifically, we consider a cosmological model in which matter is described by two scalar matter fields, one being the inflaton and the other one representing a matter field which is used as a clock. We analyze back-reaction in a phase of inflaton-driven slow-roll inflation, and find that the leading infrared back-reaction terms contributing to the evolution of the expansion rate do not vanish when measured at a fixed value of the clock field. We also analyze the back-reaction of entropy modes in a specific cosmological model with negative square mass for the entropy field and find that back-reaction can become significant. Our work provides evidence that, in general, the back-reaction of infrared fluctuations could be locally observable.

I. INTRODUCTION

Because of the nonlinear nature of the Einstein equations, the ansatz for metric and matter used all the time in early Universe cosmology, namely a homogeneous and isotropic background space-time plus fluctuations which average to zero on the spatial hypersurfaces (set by the background cosmology) and which obey the linear fluctuation equations, does not obey the Einstein equations at second order in the expansion parameter (the fractional amplitude of the linear fluctuations). Terms which have to be added to the metric such that the Einstein equations are satisfied to second order are called *back-reaction terms*.

The back-reaction of short wavelength gravitational waves on an expanding Friedmann-Robertson-Walker (FRW) cosmology is a well-established subject [1]. We insert the ansatz for the metric consisting of linear fluctuations $\delta g_{\mu\nu}$ about the FRW background $g_{\mu\nu}^0$

$$g_{\mu\nu} = g_{\mu\nu}^0 + \delta g_{\mu\nu} \quad (1)$$

into the Einstein equations

$$G_{\mu\nu} = T_{\mu\nu}, \quad (2)$$

$G_{\mu\nu}$ and $T_{\mu\nu}$ denoting the Einstein tensor and the energy-momentum tensor, respectively, expand to second order, use the fact that the linear fluctuation equations are satisfied to cancel the linear terms, and move all quadratic terms to the right hand side of the equation. These quadratic terms are interpreted as an effective energy-momentum tensor $\tau_{\mu\nu}$ (“pseudotensor”) for fluctuations. The resulting equation

$$G_{\mu\nu}(g^{br}) = T_{\mu\nu}(\varphi^{br}) + \tau_{\mu\nu} \quad (3)$$

(up to terms of third and higher order) can be interpreted in the sense that the fluctuations carry energy

and pressure and hence effect the background space-time in which they are defined. Note, that in order for this equation to be satisfied, we need to add quadratic correction terms (“back-reaction terms”) to the metric and the matter (which explains the superscripts in the above equation). In particular, the homogeneous component of the metric obtains a correction term of quadratic order which can be extracted from (3) by taking spatial averages (see [2,3] for more details).

Since, in the context of inflationary and post-inflationary cosmology, the scalar metric fluctuations (fluctuations coupled to energy density and pressure perturbations) are believed to dominate over the effects of gravitational waves, it is of great interest to study the back-reaction of these cosmological perturbations. Furthermore, in inflationary cosmology the phase space of infrared modes (defined as modes with wavelength greater than the Hubble radius) is growing over time, whereas the phase space of ultraviolet modes does not grow, and since the amplitude of the associated metric fluctuations of these infrared modes does *not* decrease in time (see e.g. [4] for a comprehensive overview of the theory of cosmological fluctuations and [5] for a recent introductory overview), the back-reaction of these infrared modes may be very important.

Rather recently, the effective energy-momentum tensor formalism of [1] was generalized to study the back-reaction of infrared modes on the spatially homogeneous component of the metric [6,2]. The surprising result which was found is that in a slow-roll inflationary background, the infrared contribution to $\tau_{\mu\nu}$ has the form of a negative cosmological constant, with the absolute value of the energy density increasing in time as the phase space of infrared modes grows and the amplitude of the gravitational potential remains large (in fact the amplitude is slowly increasing). A similar result was found in an earlier study by Tsamis and Woodard [7], in which the back-reaction of long wavelength gravitational waves

in a de Sitter background was studied ^{*}. The results of [6,2] were confirmed in [21] using very different techniques. On the basis of these calculations, it was argued [6,2,21] that gravitational back-reaction in scalar field-driven inflationary models, calculated up to quadratic order in perturbations and to leading order in the long wavelength expansion and in the slow roll approximation could decrease the expansion rate of the universe and potentially solve the cosmological constant problem [22,23] (see also [7,24] for similar suggestions in the context of back-reaction studies of gravitational waves, and [25] for earlier ideas concerning the instability of de Sitter space).

However, as was emphasized by Unruh [26] [†], the approach of [6,2,7] has several shortcomings. First of all, due to the nonlinear nature of the Einstein equations, calculating an “observable” from the spatially averaged metric will not in general give the same result as calculating the spatially averaged value of the observable. More importantly, the spatially averaged metric is not a local physical observable. An improved analysis of gravitational back-reaction on the locally measurable expansion rate of the Universe starts with identifying a local physical variable which describes this expansion rate, then calculating the back-reaction of cosmological perturbations on this quantity, and only at the end taking an expectation value of the result. It is important to fix the hypersurface of averaging by a clear physical prescription in order to remove the possibility of being misled by coordinate artifacts which are not physically measurable. Such an approach was first suggested by Abramo and Woodard in [27].

In a first paper [28], we considered a simple variable describing the local expansion rate and calculated this variable to second order in the metric fluctuations in a model with a single matter field, the inflaton field φ . We then evaluated the leading contributions of infrared fluctuation modes to this variable. When evaluated at a fixed background time t , we found that the back-reaction effects were present. However, when evaluated at a fixed value of the matter field φ , the only physical clock available in this simple system, the dependence of the expansion rate on the clock time took on exactly the same form as in an unperturbed background. Thus, the leading infrared back-reaction terms had no locally measurable effect in this system. Similar conclusions were reached in [29].

^{*}The back-reaction of small-scale (i.e. smaller than the Hubble radius) cosmological perturbations has been considered in [8–15], and in [16,17] in the context of Newtonian cosmological perturbation theory. Note also that Nambu [18–20] is developing a program to compute back-reaction effects on the spatially averaged metric using the renormalization group method.

[†]We are also grateful to Alan Guth, Andrei Linde and Alex Vilenkin for detailed private discussions on these points.

However, the fact that the infrared back-reaction terms did not vanish when evaluated at a fixed background time leads us to the conjecture that they will have a locally measurable effect provided we have a clock different from the inflaton field φ itself [‡]. The simplest way to introduce an independent physical clock is to add a second scalar matter field χ to the system. This field χ represents our observed matter fields.

In this paper, we evaluate our variable describing the local expansion rate of the Universe to second order in the fluctuations in the slow-rolling phase of an inflationary Universe with two matter field, the inflaton φ which dominates the energy-momentum tensor and a spectator (clock) field χ . We find that, in general, the leading infrared terms **do not** vanish when evaluated at a fixed value of the clock field χ . Thus, in this system infrared back-reaction is for real [§]. We also discuss some subtleties concerning this conclusion. The physical effects of this infrared back-reaction remain to be investigated [31].

The outline of this article is as follows:

In the following section we review the construction of the variable Θ which we propose to use to describe the local cosmological expansion rate, and we summarize the results of our previous paper [28] in which this variable was computed in a cosmological model with a single scalar matter field.

In Section 3, we derive the expression for Θ in a model with two matter fields in terms of the values of the metric fluctuations (more specifically, in terms of the gravitational potential Φ which characterizes the fluctuations in the longitudinal gauge).

In Section 4, we derive an expression for the change of Θ over a short time during which the clock field χ changes by a small amount, in terms of physically measurable quantities. We investigate the behavior of back-reaction in scenario where adiabatic modes are dominant and χ is a rapidly oscillating scalar field where the local energy density of χ is our clock. We show that when evaluated during the slow-roll phase of an inflationary Universe, there are non-vanishing back-reaction contributions from infrared modes.

Then, in section 5, we consider a two matter field model motivated by hybrid inflation, where it is expected that the entropy fluctuations are large. We calculate the back-reaction effect in terms of proper time, as a clock and we show that back-reaction of entropy modes could be significant.

[‡]To explain the effects of back-reaction in heuristic language one needs to introduce a second matter field: the back-reaction effect is the fact that the expansion rate is different in the presence of fluctuations compared to the result in the absence of fluctuations *when measured at the same value of the temperature of the CMB*.

[§]Another system in which infrared back-reaction is for real was discussed in [30].

In the final section we summarize and discuss our results and point out issues which remain to be resolved.

II. LOCAL BACK-REACTION IN SINGLE FIELD MODELS

We begin by reviewing the construction of the variable Θ measuring the local cosmological expansion rate: We consider the velocity four-vector field u^α tangential to a family of world lines of a matter fluid in a general space-time. This four-vector is normalized such that

$$u^\alpha u_\alpha = 1, \quad (4)$$

where α runs over the space-time indices. In terms of this four-vector, the local expansion rate Θ is defined by

$$\Theta \equiv u^\alpha_{;\alpha}. \quad (5)$$

It represents the local expansion rate of the tangential surfaces orthogonal to the fluid flow.

For a homogeneous Universe with scale factor $a(t)$, the Hubble expansion rate H is related to Θ via

$$H \equiv \partial_t a / a = \frac{1}{3} \Theta. \quad (6)$$

Thus, for a Universe with fluctuations, we use $\partial_t S / S \equiv \frac{1}{3} \Theta$ to define the “locally measured” Hubble expansion rate. This variable is a mathematically simpler object than the one recently introduced in [27], which involves the integral along the past light cone of the observation point of a function which itself is rather difficult to calculate.

In [28], we considered a theory with a single matter field φ and evaluated Θ up to second order in the metric fluctuation variable Φ (defined later in (16)). When expressed as a function of physical time t , and dropping all spatial gradient terms, the result obtained is

$$\Theta = 3 \frac{\dot{a}}{a} (1 - \Phi + \frac{3}{2} \Phi^2) - 3(\dot{\Phi} + \Phi \dot{\Phi}) \quad (7)$$

Thus, at first glance it looks like the local expansion rate is different in the presence ($\Theta \neq 0$) than in the absence ($\Theta = 0$) of fluctuations. However, the above conclusion is premature since the background time t is not a physical quantity.

In order to draw conclusions about the local measurability of the back-reaction of infrared fluctuation modes, it is necessary to express the final result in terms of a locally measurable quantity, a clock. The only such clock available in the single matter field model considered in [28] is the inflaton field itself. Thus, we must express the metric fluctuations Φ in terms of the matter fields. This can be done by making use of the Einstein constraint equations. Neglecting spatial derivatives in these equations, one can show (see also [32]) that, also setting

$\dot{\Phi} = 0$ (which is approximately the case for each long wavelength fluctuation mode in a slow-roll inflationary model [4]), the final result for the local expansion rate becomes

$$\Theta = \sqrt{3} \sqrt{V(\varphi)} \quad (8)$$

which as a function of φ is the same as the relation for an unperturbed background. Thus, in the single matter field model the leading back-reaction terms from infrared modes are not locally measurable, in agreement with the conjectures on [26] **

In the presence of a second matter field χ , it is possible to use this field as a clock. In fact, using this field to model the temperature of the CMB, it is physically more sensible to use χ as the clock rather than the inflaton field itself. The fact that in the single field matter model the dominant infrared back-reaction terms *do not* vanish when evaluated at a fixed background time t leads us to conjecture that they *will not* vanish when evaluated at a fixed value of the new clock field χ in the two matter field model.

If we are able to verify the above conjecture, we would have established a close analogy (already suggested in [28]) with the analysis of the parametric amplification of super-Hubble-scale cosmological fluctuations during inflationary reheating. From the point of view of the background space-time coordinates, it appears [34] that the parametric amplification of matter fluctuations on super-Hubble scales in an unperturbed cosmological background (see e.g. [35,36] for a discussion of parametric resonance during reheating) would imply the parametric amplification of the cosmological fluctuations on these scales. However, it can be shown that in single field models physical observables measuring the amplitude of cosmological fluctuations do not feel any resonance [37–39,32]. In contrast, in two field models of inflation there is [40,41] parametric amplification of super-Hubble-scale cosmological fluctuations. In this case, there is a fluctuation mode corresponding to entropy fluctuations which cannot locally be gauged away. This mode is (in certain theories) parametrically amplified during reheating, and in turn drives the parametric resonance of the super-Hubble scale curvature fluctuations.

III. DERIVING THE EXPANSION RATE FOR MODELS WITH TWO MATTER FIELDS

The procedure for calculating Θ will be as follows: First, we determine the velocity four-vector field u^α for

**Note, however, that there still could be back-reaction of infrared terms which can change the course of evolution of φ , due to the fact that new modes are continuously exiting the Hubble radius. These effects are calculated in [33].

the two field model. Then, we use the Einstein equations to express u^α in terms of the metric perturbations. Taking the relative amplitude of the metric fluctuations as the expansion parameter, we then calculate Θ , our local measure of the Hubble expansion rate, to second order. After evaluating the result on a physically determined hypersurface we can then study the back-reaction of cosmological fluctuations on the locally measured Hubble expansion rate. We will focus on the leading infrared contributions to back-reaction, the terms found to dominate the back-reaction effects in [2,6,21].

We consider a model with two scalar matter fields, the inflaton φ and the clock field χ :

$$\mathcal{L} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - V(\varphi, \chi) \quad (9)$$

$$\mathcal{L}' = \frac{1}{2} \partial^\mu \chi \partial_\mu \chi. \quad (10)$$

We describe the inflaton as a perfect fluid with energy density ρ and pressure P , considering

$$(\rho + P)u_\mu u_\nu - P g_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \mathcal{L} \cdot g_{\mu\nu} \quad (11)$$

which can be justified by taking

$$P = \mathcal{L} \text{ and} \quad (12)$$

$$\rho = \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi, \chi), \quad (13)$$

which leads to

$$u_\mu = A \partial_\mu \varphi \quad (14)$$

$$A = (\partial^\nu \varphi \partial_\nu \varphi)^{-1/2}. \quad (15)$$

Starting from the expression for the metric to linear order in the fluctuations Φ (see [4] for a detailed review), we determine the velocity four-vector field u^α to second order, the order required to analyze the leading infrared terms in the back-reaction to quadratic order. In linear perturbation theory, and in the case of simple forms of matter for which there is to linear order no anisotropic stress, the metric (in longitudinal gauge) can be written as

$$ds^2 = a(\eta)^2 ((1 + 2\Phi)d\eta^2 - (1 - 2\Psi)\gamma_{ij}dx^i dx^j), \quad (16)$$

$$\gamma_{ij} = \delta_{ij} [1 + \frac{1}{4} \mathcal{K}(x^2 + y^2 + z^2)] \quad (17)$$

where $\mathcal{K} = 0, 1, -1$ depending on whether the three-dimensional space corresponding to the hypersurface $t = \text{const.}$ is flat, closed or open. In this paper we will take it to be zero in order to simplify the calculations. The time variable η appearing in (16) is conformal time and is related to the coordinate time t via $d\eta = a^{-1}dt$. For the forms of matter considered here, $\Psi = \Phi$ at linear order

^{††}. The field Φ is then called the relativistic gravitational potential.

In order to obtain the complete result for gravitational back-reaction, we consider the Einstein equations for a perfect fluid and additional scalar field χ ,

$$G_{\mu\nu} = (P + \rho)u_\mu u_\nu - P g_{\mu\nu} + \partial_\mu \chi \partial_\nu \chi - \mathcal{L}' g_{\mu\nu} \quad (18)$$

(in units in which $8\pi G = 1$), which, since $G^\mu_\mu = -R$, will yield

$$-R = \rho - 3P - \partial^\mu \chi \partial_\mu \chi. \quad (19)$$

We can define a new tensor called $G'_{\mu\nu}$ and a new scalar R' by,

$$G'_{\mu\nu} = G_{\mu\nu} - \partial_\mu \chi \partial_\nu \chi + \mathcal{L}' \cdot g_{\mu\nu} \quad (20)$$

$$R' = R - \partial^\mu \chi \partial_\mu \chi. \quad (21)$$

Then, the equations become similar to the case of a single scalar field (see [28]), and ρ satisfies the equation,

$$\rho = u^\mu G'_{\mu\nu} u^\nu \quad (22)$$

which leads to an equation that can be solved perturbatively for u_i and u_0 in parallel with equation (4) to any desired order:

$$G_i^{00} = \frac{4}{3} u^\mu G'_{\mu\nu} u^\nu u^0 u_i + R' u^0 u_i. \quad (23)$$

For an unperturbed Robertson-Walker metric, the four-velocity field u in comoving coordinates would be

$$u^\mu = (1, 0, 0, 0), \quad (24)$$

which can be substituted in equation (23) as zeroth order solution in an expansion in powers of Φ to calculate u^i up to second order,

$$u^{i(2)} = \frac{G^{0i} - \partial^0 \chi \partial^i \chi}{\frac{4}{3} u^{03} G'_{00} + \frac{1}{3} R' u^0}, \quad (25)$$

where G^{0i} is proportional to $\partial^i \Phi$. Using equation (4) we can derive the expression for the time component of u^α in terms of Φ :

$$u^0(\eta) = a^{-1} (1 - \Phi + \frac{3}{2} \Phi^2) + \frac{1}{2} a u^i u^i. \quad (26)$$

Now that we have all components of u^α , we need to take the covariant derivative of this vector and retain all Φ dependence up to second order. During inflation, fluctuations which are generated on sub-Hubble scales early on during the inflationary phase are red-shifted to scales

^{††}Even if we did not make the assumption $\Psi = \Phi$, it turns out that at second order all infrared terms depending on $\Psi - \Phi$ will drop out.

much larger than the Hubble radius. Thus, in this context it is of great interest to consider the back-reaction of infrared modes. To investigate this effect we neglect any spatial derivative term contributing to the local expansion rate Θ . Since

$$u^\alpha{}_{;\alpha} = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} u^\mu), \quad (27)$$

and, according to equation (25), spatial components of u are proportional to spatial derivatives of Φ and χ , we conclude that, neglecting the spatial derivative terms, the expression for Θ reduces to

$$\Theta = \frac{1}{\sqrt{g}} \partial_0 (\sqrt{g} u^0). \quad (28)$$

A straightforward calculation yields the result

$$\Theta = 3 \frac{\dot{a}}{a} (1 - \Phi + \frac{3}{2} \Phi^2) - 3(\dot{\Phi} + \dot{\Phi} \Phi). \quad (29)$$

Where a dot denotes differentiation with respect to the coordinate time t . Since $H = \dot{a}/a$, we can immediately read off the extra terms contributing to the local expansion rate which result from the presence of cosmological fluctuations. Hence, it follows that if evaluated at a fixed coordinate time, infrared modes on average lead to corrections in the expansion rate compared to what would be obtained at the same coordinate time in the absence of metric fluctuations. Whether this is a physically measurable effect will be discussed in more depth in the following section.

IV. EXPANSION RATE IN TERMS OF A SCALAR FIELD AS OBSERVABLE

The expression (29) gives Θ in terms of the metric fluctuations. However, to determine whether this local effect of gravitational back-reaction of cosmological fluctuations is physically measurable, we need to express Θ in terms of a physical clock field, not just in terms of the background time. As was mentioned before, we will use χ as a clock. Hence, to obtain results with physical meaning, we must evaluate Θ on a surface of constant χ and not constant t .

By dropping the gradient terms from the G^{00} Einstein equations, it can be shown that

$$\frac{H}{\sqrt{1+2\Phi}} = \frac{1}{\sqrt{3}} \left[V(\varphi, \chi) + \frac{1}{2} \frac{\dot{\varphi}^2}{g_{\phi\phi}} + \frac{1}{2} \frac{\dot{\chi}^2}{g_{\chi\chi}} \right]^{1/2} \quad (30)$$

where $(1+2\Phi)$ is simply g_{00} , the time-time component of metric. If we expand the left hand side in terms of Φ we get the result,

$$H(1 - \Phi + \frac{3}{2} \Phi^2) = \frac{1}{\sqrt{3}} \left[V(\varphi, \chi) + \frac{1}{2} \frac{\dot{\varphi}^2}{g_{\phi\phi}} + \frac{1}{2} \frac{\dot{\chi}^2}{g_{\chi\chi}} \right]^{1/2}. \quad (31)$$

Substituting this result into equation (29) and neglecting terms containing Φ (which are sub-dominant compared to other terms in the infrared limit as long as the equation of state of the background is not changing significantly over time) leads to Friedmann-like equation for the local expansion rate

$$\Theta = \sqrt{3} \left[V(\varphi, \chi) + \frac{1}{2} \frac{\dot{\varphi}^2}{g_{\phi\phi}} + \frac{1}{2} \frac{\dot{\chi}^2}{g_{\chi\chi}} \right]^{1/2}. \quad (32)$$

At this point, it appears at first glance that the expansion rate for a fixed value of the clock field manifestly depends on the value of the gravitational potential Φ which enters via g_{00} , and that thus the leading infrared back-reaction effects are locally measurable. However, we are not yet ready to reach this conclusion since the right hand side of (32) still contains derivatives with respect to the background coordinate time which should be re-expressed in terms of local physical observables.

To describe the behavior of Θ on surfaces of constant χ , we have to express the right hand side of (32) in terms of χ and of the initial values φ_{in} and χ_{in} of the matter fields. This involves solving the Klein-Gordon equations for both scalar fields:

$$V_{,\varphi}(\varphi, \chi) + 3 \frac{H}{g_{\phi\phi}} \dot{\varphi} + \frac{1}{g_{\phi\phi}} \ddot{\varphi} = 0 \quad (33)$$

$$V_{,\chi}(\varphi, \chi) + 3 \frac{H}{g_{\chi\chi}} \dot{\chi} + \frac{1}{g_{\chi\chi}} \ddot{\chi} = 0. \quad (34)$$

Since finding exact and analytical solutions of this coupled system of equations is not possible for a general form of the potential, we solved these equations for an infinitesimal period of time δt . In other words, we used solutions linear in time to investigate the dynamic behavior of Θ . We start with initial values of $\varphi_{in}(x)$, $\dot{\varphi}_{in}$, $\dot{\chi}_{in}$ at t_{in} on a surface with a constant value of χ_{in} , let the system evolve, and calculate the final value of these parameters on a surface with constant value of χ_f . The infinitesimal evolution equations are

$$\delta t = \frac{\chi_f - \chi_{in}}{\dot{\chi}_{in}} \quad (35)$$

$$\dot{\varphi}_f = \ddot{\varphi}_{in} \delta t + \dot{\varphi}_{in} \quad (36)$$

$$\varphi_f = \dot{\varphi}_{in} \delta t + \varphi_{in} \quad (37)$$

$$\dot{\chi}_f = \ddot{\chi}_{in} \delta t + \dot{\chi}_{in}. \quad (38)$$

Notice that we do not assign a uniform time to all points on the initial hypersurface, and that, for now, we just solve equations locally. Inserting the results from the above equations into the general expression (32) for Θ_f leads to

$$\Theta_f = \Theta_{in} - \frac{3(\chi_f - \chi_{in})}{2\dot{\chi}_{in}\sqrt{g_{\phi\phi}}} (\dot{\varphi}_{in}^2 + \dot{\chi}_{in}^2), \quad (39)$$

where the square root of $g_{\phi\phi}$ is evaluated on the initial hypersurface. We are still not finished, because the second term on the right hand side still contains derivatives with

respect to the coordinate time. So we rewrite the equation (39) using the proper time instead of the coordinate time ($\partial\tau = \sqrt{g_{00}}\partial t$), for gauge invariance purposes,

$$\Theta_f = \Theta_{in} - \frac{3}{2}\chi'(1 + \frac{\varphi_{in}'^2}{\chi_{in}'^2})\delta\chi \quad (40)$$

where $\delta\chi = \chi_f - \chi_{in}$. Equation (39) can be further simplified using Equations (30) and (32), and this enables us to express Θ_f as a function of Θ_{in} , φ_{in} , χ and χ' ,

$$\Theta_f = \Theta_{in} - \frac{3}{2}(\frac{\Theta_{in}^2}{3} - V(\varphi_{in}, \chi_{in}))\frac{\delta\chi}{\chi_{in}'} \quad (41)$$

This is the completely local result for the change in Θ which we were aiming to obtain. One may think that (41) implies there is no local back-reaction^{††}. However, this conclusion is not correct because in the presence of cosmological fluctuations the trajectories of the χ and φ fields start to deviate from the classical solution that are obtained from Equations (33) and (34). Thus, the values of φ'_{in} and χ'_{in} for the next time steps will fluctuate on the $\chi_{in} = \text{const}$ hypersurface, whereas in the absence of such fluctuations, the initial values at each step follow the classical trajectory. In other words, compared to the classical solution, there is a difference in the evolution of Θ due to the different paths for each point of an inhomogeneous universe in field space. Furthermore, there are other terms coming from the modes that are exiting the Hubble radius and are not taken into account in the infrared limit of Equation (41). Interpreted correctly, our result (41) thus shows that one expects that in a two matter field model, infrared cosmological fluctuations have a locally observable effect if cosmological perturbations are excited.

To clarify this point we investigate the behavior of back-reaction terms in two scenarios, first where adiabatic modes are dominant and then, in the next section, in a scenario where entropy modes play a major role. In the former case we assume that the χ field does not significantly contribute to the energy density of the universe and only plays the role of a clock. We can consider χ to be a rapidly oscillating scalar field (to mimic the cosmic microwave background radiation in our Universe) and take the local energy density of χ as our clock. For that purpose we work with the following potential

$$V(\chi) = \frac{1}{2}\mu^2\chi^2, \quad (42)$$

where, by taking $\mu \gg \Theta$, the rate of oscillation of χ will be much faster than the expansion rate, which enables us to solve Equation (34) using the WKB approximation and to calculate the value of $\rho_\chi = \chi'^2/2 + V(\chi)$, the energy density of χ ,

$$\rho_\chi = \rho_{\chi,in} \exp(-\int_{\tau_{in}}^{\tau} \Theta d\tau). \quad (43)$$

For the inflaton field, we also take a standard quadratic potential of chaotic inflation,

$$V(\varphi) = \frac{1}{2}m^2\varphi^2. \quad (44)$$

To take into account the effects of the ultraviolet modes that become infrared as the inflationary comoving Hubble radius shrinks, we modify Equation (33) in the slow-roll regime to obtain

$$\Theta\varphi' + m^2\varphi = C_a\Theta^{5/2}\xi(\mathbf{x}, \tau), \quad (45)$$

where C_a (a standing for adiabatic) is a numerical constant and ξ is a Gaussian random function with the covariance

$$\langle \xi(\mathbf{x}_1, \tau_1)\xi(\mathbf{x}_2, \tau_2) \rangle = \delta(\tau_1 - \tau_2)\text{sinc}(aH|\mathbf{x}_1 - \mathbf{x}_2|). \quad (46)$$

In order to derive Equation (45) rigorously, one needs to coarse-grain the non-linear Einstein equations on the scale of the Hubble radius^{§§}. The stochastic term on the right hand side of Equation 45 is due to the adiabatic fluctuation modes that are exiting the Hubble radius (see [33] for a detailed derivation).

Now, using the fact that Equation (32) simplifies to $\Theta = \sqrt{3V(\varphi)}$ in the slow-roll approximation, one can solve Equation (45) perturbatively in ξ , (see [33] for details) and calculate the expectation value of Θ ,

$$\Theta \simeq \langle \Theta \rangle = m^2(-\tau) + \frac{C_a^2}{3\sqrt{6}}(\Theta_{in}^3 - m^6(-\tau)^3), \quad (47)$$

where we have taken τ to be negative during and to go to zero at the end of the slow-roll period. This result shows that back-reaction increases the expansion rate when proper time is used as the clock. As we shall see, this result changes when we use ρ_χ as a clock. To simplify the expression for the expansion rate when ρ_χ is used as a clock, we define a new parameter, $\beta = \ln \frac{\rho_\chi}{\rho_{fc}}$, which is a measure of ρ_χ . Here, ρ_{fc} is a constant and the

^{††}The completely local form of Equation (41) is easy to understand from the mathematical point of view: by neglecting all of the spatial derivative terms, not only are we treating our equations completely locally, but we are also eliminating all interactions between neighboring points in space. Thus, the final result for Θ at a specific point in space can depend on nothing more than on the initial values of the fields φ_{in} and Θ_{in} at that same point. In other words, the dynamics of the fields at each point of space behaves like a Friedmann Universe with the appropriate initial conditions.

^{§§}This is similar to the approach taken in deriving the equations of the stochastic scenario of inflation [42]. A more rigorous derivation which includes metric perturbations is given in [32].

subscript fc refers to the time $\tau_{fc} = 0$, the end of the classic slow-roll inflation. Hence, we have

$$\beta(\tau) = - \int_0^\tau \Theta(\tilde{\tau}) d\tilde{\tau}. \quad (48)$$

Now, substituting Θ from Equation (47), in the above integral we can calculate τ in terms of β ,

$$-\tau \simeq \frac{\sqrt{2\beta}}{m} \left\{ 1 - \frac{C_a^2}{2\beta\sqrt{6}} \left(\frac{1}{3} m^4 \beta^2 + \frac{\Theta_{in}^3 \sqrt{2\beta}}{3m} \right) \right\}. \quad (49)$$

Substituting τ back in Equation (47), we can drive Θ in terms of β :

$$\Theta \simeq m\sqrt{2\beta} - \frac{13C_a^2}{24\sqrt{3}} m^3 (\beta)^{3/2} \quad (50)$$

Interestingly, the sign of the back-reaction term is negative, whereas it was positive in Equation (47). Thus, it appears that when we measure local expansion rate using an oscillating field as our clock, unlike when we used proper time, back-reaction slows down the expansion rate. Describing the back-reaction in terms of β is of more relevance to the inflationary predictions for today's universe, as β is proportional to number of e-foldings of inflation which enters into the calculation of the power spectrum of metric fluctuations.

V. BACK-REACTION OF ENTROPY MODES

In this section we will consider a two matter field model motivated by hybrid inflation [43], where it is expected that the entropy fluctuations are large. In this section, due to the specific choice of the model that we investigate, we calculate the back-reaction effect in terms of proper time, i.e. constant- τ surfaces, as a clock instead of constant- χ surfaces. This is an appropriate choice because we only want to investigate the magnitude of this effect. In terms of this clock variable, Equation (41) takes the following form,

$$\begin{aligned} \Theta_f &= \Theta_{in} - \frac{3}{2} (\chi_{in}'^2 + \varphi_{in}'^2) \delta\tau \\ &= \Theta_{in} - \frac{3}{2} \left(\frac{\Theta_{in}^2}{3} - V(\varphi_{in}, \chi_{in}) \right) \delta\tau \end{aligned} \quad (51)$$

where we see that different trajectories in the (φ, χ) space, which could belong to different points in an inhomogeneous universe, lead to different behaviors for Θ . As we demonstrate below, this effect could be clearly seen when we consider the evolution of Θ in the long run.

A simple model where we get growing entropy modes is hybrid inflation, where the secondary field, χ , has a negative square mass. In the slow-roll regime, the classical value of χ vanishes, which is why we do not use it as our clock. Now, notice that the local evolution of Θ is given by

$$\Theta_f = \Theta_{in} - \frac{3}{2} \int_{\tau_i}^{\tau_f} (\chi'^2 + \varphi'^2) d\tau. \quad (52)$$

If we can show that, due to the presence of entropy modes, χ'^2 may dominate φ'^2 in Equation (52), we have demonstrated that back-reaction can dominate the evolution of Θ .

For the sake of clarity, let us investigate the behavior of χ'^2 in a model where,

$$V(\varphi, \chi) = \frac{1}{2} m^2 \varphi^2 - \frac{1}{2} \mu^2 \chi^2. \quad (53)$$

If we assume χ is also slowly rolling, then, similar to what happens in Equation (45), the field equation (34) should be modified to take into account the entropy fluctuation modes, while we ignore such modifications (arising due to the adiabatic modes) in Equation (33). This leaves us with the following field equations

$$\Theta \chi' - \mu^2 \chi = C_e \Theta^{5/2} \xi(\mathbf{x}, \tau), \quad (54)$$

$$\Theta \varphi' + m^2 \varphi = 0, \quad (55)$$

where C_e (e standing for entropy) is a numerical constant. Again, using $\Theta \simeq \sqrt{3V(\varphi)}$, we can calculate χ up to first order in ξ ,

$$\chi \simeq C_e m^3 (-\tau)^{1-\mu^2/m^2} \int_{\tau_{in}}^{\tau} (-\tilde{\tau})^{3/2+\mu^2/m^2} \xi(\mathbf{x}, \tilde{\tau}) d\tilde{\tau}, \quad (56)$$

which yields

$$\langle \chi'^2 \rangle = f(\tau) + \frac{C_e^2 \mu^4 m^2}{4 + 2\frac{\mu^2}{m^2}} \left(\frac{\tau_{in}}{\tau} \right)^{2\mu^2/m^2} \tau_{in}^4, \quad (57)$$

where $f(\tau)$ is a correction due to the fluctuations of the Hubble size modes, and its amplitude decreases with time. For slow-roll conditions to be satisfied, we need to have $\tau > \frac{1}{m}$. Inserting this limit for the growing mode in $\langle \chi'^2 \rangle$ and comparing it with homogeneous value of φ'^2 , which is the classical rate of change in Θ , we get

$$\frac{\langle \chi'^2 \rangle}{\varphi'^2} \simeq C_e^2 \frac{\mu^4}{4m^4 + 2\mu^2 m^2} \left(\frac{3}{2} \right)^{2+\mu^2/m^2} \varphi_{in}^{4+2\mu^2/m^2}, \quad (58)$$

at the end of slow-roll inflation. Since $\varphi_{in} > 1$, this relation implies that if $\mu \gtrsim m$, then back-reaction dominates the inflation before its end (calculated in the absence of back-reaction), and the homogenous background solution loses its validity.

This result can also be important for calculating the spectrum of fluctuation after inflation ends. We have also seen that since different points have different initial conditions, they will have different final expansion rates. To obtain a quantitative measure of the back-reaction of infrared modes, one needs to take an average of the expansion rate over the many initial Hubble patches that form our observable universe.

VI. CONCLUSIONS

In this paper, we have studied the leading back-reaction effects of long wavelength cosmological fluctuations on a local observable Θ which measures the local expansion rate of the Universe. The observable gives the rate at which neighboring comoving observers separate and coincides with the usual definition of expansion in the context of the fluid approach to cosmology. We considered models with two matter fields, an inflaton field φ which dominates the energy-momentum tensor and leads to slow-roll inflation, and a second matter field χ which was used either as a clock or it contributed to production of entropy modes. We investigated the behavior of back-reaction terms in two scenarios, first where adiabatic modes are dominant and then, in a scenario where entropy modes play a major role. In the former case to obtain locally measurable statements about the presence or absence of back-reaction effects, We considered χ to be a rapidly oscillating scalar field (to mimic the cosmic microwave background radiation in our Universe) and took the local energy density of χ as our clock. We evaluated Θ and found that not only the back-reaction term does not vanish but the sign of the back-reaction term is negative as well. In the second scenario, we analyzed the back-reaction of entropy modes in a specific cosmological model with negative square mass for the entropy field and found that back-reaction can become significant and dominate the evolution of Θ .

Our main result is that the leading infrared terms, the terms which dominate the effects discussed in [2] and [21], are non-vanishing and physically measurable if we use the second scalar field χ as a clock. The effect can be particularly large if the entropy mode of the cosmological fluctuation field is excited. This confirms the conjectures made in our previous work [28] and is in good agreement with the physical intuition gained from the study of the parametric amplification of super-Hubble length cosmological fluctuations during reheating [40,41].

In this paper, we have thus shown that infrared back-reaction is “for real” in two matter field models. However, how important the back-reaction is remains to be investigated. Results will be reported in a subsequent publication [31].

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